4. $f(x,y) = 4x + 6y$, $g(x,y) = x^2 + y^2 = 13 \quad \Rightarrow \quad \nabla f = \langle 4, 6 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2\lambda x = 4$ and $2\lambda y = 6$ imply $x = \dfrac{2}{\lambda}$ and $y = \dfrac{3}{\lambda}$. But $13 = x^2 + y^2 = \left(\dfrac{2}{\lambda}\right)^2 + \left(\dfrac{3}{\lambda}\right)^2 \quad \Rightarrow \quad 13 = \dfrac{13}{\lambda^2} \quad \Rightarrow \quad \lambda = \pm 1$, so f has possible extreme values at the points $(2,3)$, $(-2,-3)$. We compute $f(2,3) = 26$ and $f(-2,-3) = -26$, so the maximum value of f on $x^2 + y^2 = 13$ is $f(2,3) = 26$ and the minimum value is $f(-2,-3) = -26$.

6. $f(x,y)=x^2+y^2$, $g(x,y)=x^4+y^4=1 \Rightarrow \nabla f=\langle 2x,2y\rangle$, $\lambda\nabla g=\langle 4\lambda x^3,4\lambda y^3\rangle$. Then $x=2\lambda x^3$ implies $x=0$ or $\lambda=\dfrac{1}{2x^2}$. If $x=0$, then $x^4+y^4=1$ implies $y=\pm 1$. But $y=2\lambda y^3$ implies $y=0$ so $x=\pm 1$ or $\lambda=\dfrac{1}{2y^2}$ and $x^2=y^2$ and $2x^4=1$ so $x=\pm\frac{1}{\sqrt[4]{2}}$. Hence the possible points are $(0,\pm 1)$, $(\pm 1,0)$, $\left(\pm\frac{1}{\sqrt[4]{2}},\pm\frac{1}{\sqrt[4]{2}}\right)$, with the maximum value of f on $x^4+y^4=1$ being $f\left(\pm\frac{1}{\sqrt[4]{2}},\pm\frac{1}{\sqrt[4]{2}}\right)=\frac{2}{\sqrt{2}}=\sqrt{2}$ and the minimum value being $f(0,\pm 1)=f(\pm 1,0)=1$.

8. $f(x,y,z) = 8x - 4z$, $g(x,y,z) = x^2 + 10y^2 + z^2 = 5 \quad \Rightarrow \quad \nabla f = \langle 8, 0, -4 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 20\lambda y, 2\lambda z \rangle$.

Then $2\lambda x = 8$, $20\lambda y = 0$, $2\lambda z = -4$ imply $x = \dfrac{4}{\lambda}$, $y = 0$, and $z = -\dfrac{2}{\lambda}$. But

$5 = x^2 + 10y^2 + z^2 = \left(\dfrac{4}{\lambda}\right)^2 + 10\,(0)^2 + \left(-\dfrac{2}{\lambda}\right)^2 \quad \Rightarrow \quad 5 = \dfrac{20}{\lambda^2} \quad \Rightarrow \quad \lambda = \pm 2$, so f has possible extreme values at the points $(2, 0, -1)$, $(-2, 0, 1)$. The maximum of f on $x^2 + 10y^2 + z^2 = 5$ is $f(2, 0, -1) = 20$, and the minimum is $f(-2, 0, 1) = -20$.

to $y^2 + z^2 = 1$.

18. $f(x, y) = 2x^2 + 3y^2 - 4x - 5 \Rightarrow \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \Rightarrow x = 1, y = 0$. Thus $(1, 0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x, y) = x^2 + y^2 = 16 \Rightarrow \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$, so $6y = 2\lambda y \Rightarrow$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \Rightarrow x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1, 0) = -7$, $f(4, 0) = 11$, $f(-4, 0) = 43$, and $f(-2, \pm 2\sqrt{3}) = 47$. Thus the maximum value of $f(x, y)$ on the disk $x^2 + y^2 \le 16$ is $f(-2, \pm 2\sqrt{3}) = 47$, and the minimum value is $f(1, 0) = -7$.

21. $P(L,K)=bL^{\alpha}K^{1-\alpha}$, $g(L,K)=mL+nK=p \quad\Rightarrow\quad \nabla P=\langle \alpha bL^{\alpha-1}K^{1-\alpha},(1-\alpha)bL^{\alpha}K^{-\alpha}\rangle$,

$\lambda\nabla g=\langle\lambda m,\lambda n\rangle$. Then $\alpha b(K/L)^{1-\alpha}=\lambda m$ and $(1-\alpha)b(L/K)^{\alpha}=\lambda n$ and $mL+nK=p$, so

$\alpha b(K/L)^{1-\alpha}/m=(1-\alpha)b(L/K)^{\alpha}/n$ or $n\alpha/[m(1-\alpha)]=(L/K)^{\alpha(L/K)1-\alpha}$ or $L=Kn\alpha/[m(1-\alpha)]$.

Substituting into $mL+nK=p$ gives $K=(1-\alpha)p/n$ and $L=\alpha p/m$ for the maximum production.

22. $C(L,K)=mL+nK$, $g(L,K)=bL^{\alpha}K^{1-\alpha}=Q \quad\Rightarrow\quad \nabla C=\langle m,n\rangle$,

$\lambda\nabla g=\langle\lambda\alpha bL^{\alpha-1}K^{1-\alpha},\lambda(1-\alpha)bL^{\alpha}K^{-\alpha}\rangle$. Then $\dfrac{m}{\alpha b}\left(\dfrac{L}{K}\right)^{1-\alpha}=\dfrac{n}{(1-\alpha)b}\left(\dfrac{K}{L}\right)^{\alpha}$ and $bL^{\alpha}K^{1-\alpha}=Q$

$\Rightarrow\quad \dfrac{n\alpha}{m(1-\alpha)}=\left(\dfrac{L}{K}\right)^{1-\alpha}\left(\dfrac{L}{K}\right)^{\alpha}\quad\Rightarrow\quad L=\dfrac{Kn\alpha}{m(1-\alpha)}$ and so $b\left[\dfrac{Kn\alpha}{m(1-\alpha)}\right]^{\alpha}K^{1-\alpha}=Q$. Hence

$K=\dfrac{Q}{b\,(n\alpha/[m(1-\alpha)])^{\alpha}}=\dfrac{Qm^{\alpha}(1-\alpha)^{\alpha}}{bn^{\alpha}\alpha^{\alpha}}$ and $L=\dfrac{Qm^{\alpha-1}(1-\alpha)^{\alpha-1}}{bn^{\alpha-1}\alpha^{\alpha-1}}=\dfrac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}}$ minimizes

cost.

28. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x^2y^2z = 1 \Rightarrow$ $\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle 2\lambda xy^2z, 2\lambda x^2yz, \lambda x^2y^2 \rangle$. Then $\lambda y^2 z = 1$, $\lambda x^2 z = 1$ and $\lambda x^2y^2 = 2z$ so $y^2z = x^2z$ and $x = \pm y$. Also $2z/1 = \lambda x^2y^2/(\lambda x^2 z)$ so $2z^2 = y^2$ and $y = \pm\sqrt{2}\,z$. But $x^2y^2z = 1$ implies $z > 0$ and $4z^5 = 1$. Thus the points are $\left(\pm 2^{1/10}, \pm 2^{1/10}, 2^{-2/5}\right)$, and the minimum distance is attained at each of these.

41. (a) We wish to maximize $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$ subject to $g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n = c$ and $x_i > 0$.

$$\nabla f = \left\langle \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n), \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n), \ldots, \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) \right\rangle$$

and $\lambda \nabla g = \langle \lambda, \lambda, \ldots, \lambda \rangle$, so we need to solve the system of equations

$$\frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n) = \lambda \quad \Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_1$$

$$\frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n) = \lambda \quad \Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_2$$

$$\vdots$$

$$\frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) = \lambda \quad \Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_n$$

This implies $n\lambda x_1 = n\lambda x_2 = \cdots = n\lambda x_n$. Note $\lambda \neq 0$, otherwise we can't have all $x_i > 0$. Thus $x_1 = x_2 = \cdots = x_n$. But $x_1 + x_2 + \cdots + x_n = c \quad \Rightarrow \quad nx_1 = c \quad \Rightarrow \quad x_1 = \frac{c}{n} = x_2 = x_3 = \cdots = x_n$.

Then the only point where f can have an extreme value is $\left(\frac{c}{n}, \frac{c}{n}, \ldots, \frac{c}{n}\right)$. Since we can choose values for $(x_1, x_2, \ldots, x_n)$ that make f as close to zero (but not equal) as we like, f has no minimum value. Thus the maximum value is $f\left(\frac{c}{n}, \frac{c}{n}, \ldots, \frac{c}{n}\right) = \sqrt[n]{\frac{c}{n} \cdot \frac{c}{n} \cdot \cdots \cdot \frac{c}{n}} = \frac{c}{n}$.

(b) From part (a), $\frac{c}{n}$ is the maximum value of f. Thus $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{c}{n}$. But $x_1 + x_2 + \cdots + x_n = c$, so $\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$. These two means are equal when f attains its maximum value $\frac{c}{n}$, but this can occur only at the point $\left(\frac{c}{n}, \frac{c}{n}, \ldots, \frac{c}{n}\right)$ we found in part (a). So the means are equal only when $x_1 = x_2 = x_3 = \cdots = x_n = \frac{c}{n}$.